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Non-relativistic reduction of a nuclear equation of state model

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Chapter 1

Introduction

The aim of this chapter is to introduce briefly the nuclear interaction. At first, we will describe the general features of the interaction, then we will present its microscopical description and the nuclear mean-field. We will proceed explaining the relativistic mean-field theory and its non-relativistic reduction.

1.1 The Nucleon-Nucleon force

The nuclear force is characterized by some properties, which can be inferred from the nuclear phenomenology:

- **Short range:** at short distances ($r \simeq 1.5$ fm [1]), the nuclear interaction is stronger than the Coulomb force (it can overcome the Coulomb repulsion of protons in the nucleus). At long distances (approximately at lengths of the order of the atomic size), however, the nuclear force becomes feeble, and the interaction among nuclei can be described with the Coulomb interaction alone. [2]
- **Charge independence** (or isotropic invariance): the nucleon-nucleon force seems to be nearly independent of whether the nucleon is a proton or a neutron: the force between two protons is equal to the one between two neutrons. [2] [3]

- **Spin dependence:** a system of two nucleons can be in three different states, neutron-neutron (nn), proton-proton (pp) or proton-neutron (pn , which is called deuteron). However, only the ground state of the deuteron is a spin triplet bound state (and all the excited spin singlet states are not bound). This fact seems in contradiction with the charge independence which characterizes the nuclear force, but can be explained if we consider the spin alignment and the Pauli principle: two identical particles can not occupy the same quantum state. In order to follow this principle, the total wave function of the system (given by the product of a spatial and a spin component) must be antisymmetric. The ground state, however, is characterized by a symmetric spatial wave function, therefore, in order to have an antisymmetric total wave function, the spin component must also be antisymmetric, but singlet spin states are unbound. [2]
- **Noncentral:** The deuteron has a quadrupole moment (i.e. is not spherical), therefore a tensor component is needed to describe the interaction. [1]

1.1.1 The microscopic Nucleon-Nucleon interaction

We now consider the interaction in a microscopic picture.

The interaction between particles can be described thanks to other particles (which are called mesons) that act like mediators: mesons carry the interaction by transferring the energy and the quantum numbers associated with it. We can describe this kind of interaction thanks to field theory. It associates fields with the interacting particles and with the mediators.

To describe the nuclear interaction we will not consider nucleons as a system of quarks and gluons: we will consider them as point-like particles and the interaction as mediated by mesons. [1]

The oldest attempt to describe the nuclear interaction with the meson exchange model is due to Yukawa. Now, thanks to QCD, the meson theory is not perceived as fundamental any more, but still provides a good method to describe the nucleon-nucleon interaction. [4]

1.2 Mean-field models

The description of the nuclear force we made before was based on the concept of a system of two free nucleons interacting with each other. The description of a system composed by more than two nucleons is much more difficult.

The simpler way to describe the interaction between a large number of nucleon are the so-called "Mean-field models". They consist in considering a system of N non-interacting point-like particles, where each particle is moving in an external potential created by the other $N - 1$ particles. [5] [6]

The reason we can describe the interaction through a mean field description relies in the structure of the nucleus. [1] The mean free path of a nucleon in a nucleus is of the order of the nuclear radius, and it appeared difficult to accommodate this evidence into the framework of the nuclear interaction, which is strong in a shorter range than the nuclear size. The mean free path length is quite large, in fact, we know that the nuclear interaction range is of the order of approximately 1.5 fm [1]. The nucleus radius is approximately given by

$$R_N = r_0 A^{\frac{1}{3}} \quad (1.1)$$

(where $r_0 \simeq 1.25\text{fm}$ [2] and A is the atomic mass number of the nucleus), and the average distance between nucleons is of the order of 2 fm [1].

The main reason of this behavior is that the nucleons in nuclear matter and nuclei at normal density are delocalized and form a quantum liquid. This can be easily understood thanks to the relation between the zero point kinetic energy and the potential energy. [7]

The nucleons, therefore, remain at a quite large distance from one another. Also, the experiments show that, because the interaction is short range, the density remains approximately constant within the nucleus (about 0.16 fm^{-3}).

We can expect that each nucleon feels a potential which is the average of the interaction with the others. This potential corresponds to the one we considered when we performed the mean field approximation to study the nuclear equation of state. At the edge of the nucleus the density changes, thus the potential (whih is a function of the density) changes as well. In

Fig.1.1 we can see the trend of the density inside nuclei.

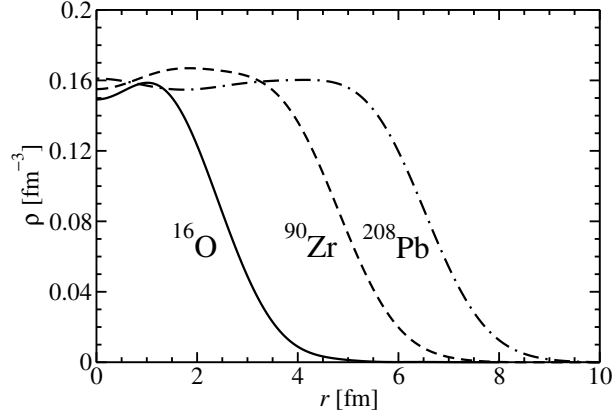


Figure 1.1: Nuclear density profiles for O16, Zr90 and Pb208.

1.2.1 Relativistic Mean-field model

The relativistic mean-field model is a phenomenological description of nuclei which consists in describing the nucleus as a system of Dirac nucleons interacting via meson fields, which constitutes a reasonable approximation to study the static properties of nuclei [8]. This model permits to describe easily many aspects of nuclear structure. For example, one can obtain easily the nuclear saturation [9]. We will consider the Hartree mean field approximation, considering the σ , ω , ρ and δ mesons as the mediators of the interaction. However, we will not consider the pion because of its negative parity ($\Pi = -1$): we will consider, in fact, only mesons with good parity, whose ground state expectation value does not vanish in the mean field approximation. [8]

We will also perform a non relativistic reduction of the model in order to understand better the relativistic and non-relativistic approaches.

Chapter 2

Solution of the Dirac equation for a free particle

The Hamiltonian for a Dirac particle of mass m takes the form [10]

$$\hat{H} = \vec{\alpha} \cdot \vec{P} + \hat{\beta}m \quad (2.1)$$

where $\vec{P}$ is the momentum operator for the particle and $\vec{\alpha}$ and $\hat{\beta}$ represent the Dirac matrices:

$$\vec{\alpha} = \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\beta_i = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

where $\sigma_i, i = x, y, z$ are the Pauli matrices and $\mathbb{1}$ is the identity.

In the coordinate base, we have $\vec{P} = -i\hbar\vec{\nabla}$, thus the Dirac equation for a free particle becomes:

$$\left[-i\hbar \vec{\alpha} \cdot \vec{\nabla} + \hat{\beta}m \right] \psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) \quad (2.2)$$

We now assume the wave function to be time independent (i.e. $\psi(\vec{r}, t) = \psi(\vec{r})$), so we look for stationary states. The equations becomes:

$$\left[-i\hbar \vec{\alpha} \cdot \vec{\nabla} + \hat{\beta}m \right] \psi(\vec{r}) = 0 \quad (2.3)$$

Since the Hamiltonian is only a function of $\vec{P}$, then $[\vec{P}, \hat{H}] = 0$, so that the eigenvalues $\vec{p}$ of $\vec{P}$ can be used to characterize the states. We look in particular for free particle solutions of the form

$$\psi_p(\vec{r}) = u_p e^{\frac{i\vec{p} \cdot \vec{r}}{\hbar}}$$

where u_p is a 4-components vector which satisfies the equation:

$$\left[\vec{\alpha} \cdot \vec{p} + \hat{\beta}m \right] u_p = 0 \quad (2.4)$$

The matrix on the left can be expressed in terms of 2×2 blocks. We therefore look for u_p in the form of a vector composed of two two-components vectors:

$$u_p = \begin{pmatrix} \phi_p \\ \chi_p \end{pmatrix}$$

We now define the Pauli vector $\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$, and we find:

$$\begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} \phi_p \\ \chi_p \end{pmatrix} = 0$$

which brings to the system of equations:

$$\begin{cases} m\phi_p + \vec{\sigma} \cdot \vec{p} \chi_p = 0 \\ \vec{\sigma} \cdot \vec{p} \phi_p - m\chi_p = 0 \end{cases} \quad (2.5)$$

whose solutions are:

$$\chi_p = \frac{\vec{\sigma} \cdot \vec{p}}{m} \phi_p \quad (2.6)$$

and

$$\phi_p = -\frac{\vec{\sigma} \cdot \vec{p}}{m} \chi_p \quad (2.7)$$

By using (2.6) in the first of equations (2.5), we obtain:

$$m^2 \phi_p + (\vec{\sigma} \cdot \vec{p})^2 \phi_p = 0$$

We know that $(\vec{\sigma} \cdot \vec{p})^2 = \vec{p} \cdot \vec{p} + i\vec{\sigma} \cdot (\vec{p} \times \vec{p}) = p^2$, thus:

$$(m^2 + p^2) \phi_p = 0$$

which means $p^2 = -m^2$, since $\phi_p \neq 0$.

We now consider positive energy states, and we choose:

$$\phi_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which means:

$$\chi_p = \begin{pmatrix} \frac{p_z}{m} \\ \frac{(p_x + ip_y)}{m} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{(p_x - ip_y)}{m} \\ -\frac{p_z}{m} \end{pmatrix}$$

We therefore find the solutions for $u_p^{(+)}$ for positive energy states:

$$u_p^{(+)} = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{m} \\ \frac{(p_x + ip_y)}{m} \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ \frac{(p_x - ip_y)}{m} \\ -\frac{p_z}{m} \end{pmatrix}$$

Considering now negative energy states, we choose

$$\chi_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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Exploiting the same procedure we used for positive energy states, we find the solutions $u_p^{(-)}$

$$u_p^{(-)} = \begin{pmatrix} -\frac{p_z}{m} \\ -\frac{(p_x + ip_y)}{m} \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\frac{(p_x - ip_y)}{m} \\ \frac{p_z}{m} \\ 0 \\ 1 \end{pmatrix}$$

Solutions corresponding to negative energy states describe antiparticles, and the components ϕ_p and χ_p of u_p describe respectively particle and antiparticle components.

Chapter 3

Relativistic hadron field theory

3.1 Lagrangian and equations of motion

In order to find the nucleon equations of motion in nuclei within a relativistic framework, we need to start from a Lagrangian ([11] [9]).

The relativistic formulation does not allow us to use the concept of an instantaneous force between nucleons. Given the wave function Ψ representing neutrons and protons, we can describe the effective nuclear interaction as mediated by meson fields, which are independent degrees of freedom of the system. We will consider only scalar ($J = 0$, where J is the internal angular momentum) or vector ($J = 1$) meson fields and isoscalar ($T = 0$, where T represents isospin) or isovector ($T = 1$) meson fields.

Furthermore, we will consider only fields with natural parity $\Pi = (-1)^J$, because we will work with nuclear states with good parity, thus fields with unnatural parity have vanishing expectation values.

Therefore, we will consider only the fields Φ_σ , $A_\mu^{(\omega)}$, $\mathbf{A}_\mu^{(\rho)}$ and Φ_δ , representing the four mesons (σ , ω , ρ and δ), where:

- Φ_σ is an isoscalar-scalar field which mediates the medium range attraction between nucleons;
- $A_\mu^{(\omega)}$ is an isoscalar-vector field which mediates a short range repulsion;
- $\mathbf{A}_\mu^{(\rho)}$ is an isovector-vector short range field which allows one to describe

asymmetric system;

- Φ_δ is an isovector- scalar field, it mediates a short range interaction and is important in order to describe asymmetric systems.

The subscript μ represents the components of the 4 - vectors $A_\mu^{(\omega)}$ and $\mathbf{A}_\mu^{(\rho)}$ (for example, $A_\mu^{(\omega)} = (A_0^{(\omega)}, A_1^{(\omega)}, A_2^{(\omega)}, A_3^{(\omega)})$ where the 0 - th component is the time-like one and the other three are space-like), while the boldface will indicate a vector in the isospin space.

We will not consider the photon field, which is responsible for the electromagnetic interaction, because we will be working in the infinite matter approximation, which consists in excluding all the surface terms and the coulomb terms.

Thus, the Lagrangian reads:

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_M + \mathcal{L}_{int} \quad (3.1)$$

where $\mathcal{L}_N$ and $\mathcal{L}_M$ are respectively the nucleonic and mesonic free Lagrangians and $\mathcal{L}_{int}$ describes the interaction between nucleons.

We can write these terms as follows:

$$\mathcal{L}_N = \bar{\Psi}(i\gamma_\mu\partial^\mu - M)\Psi \quad (3.2)$$

$$\begin{aligned} \mathcal{L}_M = & \frac{1}{2} \sum_{i=\sigma,\delta} (\partial_\mu\Phi_i\partial^\mu\Phi_i - m_i^2\Phi_i^2) \\ & - \frac{1}{2} \sum_{j=\omega,\rho} \left(\frac{1}{2} F_{\mu\nu}^{(j)} F^{(j)\mu\nu} - m_j^2 A_\mu^{(j)} A^{(j)\mu} \right) \end{aligned} \quad (3.3)$$

$$\begin{aligned} \mathcal{L}_{int} = & \bar{\Psi}\Gamma_\sigma\Psi\Phi_\sigma + \bar{\Psi}\Gamma_\delta\boldsymbol{\tau}\Psi\boldsymbol{\Phi}_\delta \\ & - \bar{\Psi}\Gamma_\omega\gamma_\mu\Psi A^{(\omega)\mu} - \bar{\Psi}\Gamma_\rho\gamma_\mu\boldsymbol{\tau}\Psi\mathbf{A}^{(\rho)\mu} \end{aligned} \quad (3.4)$$

where M is the nucleon mass, γ_μ represents the Dirac matrices ($\gamma_0 = \beta, \gamma_i = \vec{\alpha}$, where $i = 1, 2, 3$), $\boldsymbol{\tau}$ indicates the Pauli matrices in the isospin space, Γ_k

are coupling constants for $k = \sigma, \omega, \rho$ and δ and $F^{(j)}_{\mu\nu}$ is the field strength tensor for the vector fields, defined as:

$$F^{(j)}_{\mu\nu} = \partial_\mu A^{(j)}_\nu - \partial_\nu A^{(j)}_\mu \quad (3.5)$$

We now want to find the equations of motion of nucleons and mesons. In order to do so, we calculate the variational derivatives of the Lagrangian:

$$\delta S = \int d^4x \left(\frac{\delta \mathcal{L}}{\delta \bar{\Psi}} \right) = 0$$

where δS represents the variation of action. One can therefore find the equations of motion by minimizing the action S .

3.1.1 Nucleons

For nucleons we find the Dirac equation

$$[\gamma_\mu (i\partial^\mu - \hat{\Sigma}^\mu) - M^*] \Psi = 0 \quad (3.6)$$

where

$$\hat{\Sigma}^\mu = \Gamma_\omega A^{(\omega)\mu} + \Gamma_\rho \boldsymbol{\tau} \mathbf{A}^{(\rho)\mu} \quad (3.7)$$

$$M^* = M - \hat{\Sigma}^s \quad (3.8)$$

$$\hat{\Sigma}^s = \Gamma_\sigma \Phi_\sigma + \Gamma_\delta \boldsymbol{\tau} \Phi_\delta \quad (3.9)$$

represent respectively the vector self-energy, the Dirac effective mass of the nucleon and the scalar self-energy.

3.1.2 Mesons

ω **meson** The lagrangian terms in which the ω meson appears are:

$$-\frac{1}{2} \left(\frac{1}{2} F_{\mu\nu}^{(\omega)} F^{(\omega)\mu\nu} - m_\omega^2 A_\mu^{(\omega)} A^{(\omega)\mu} \right) - \bar{\Psi} \Gamma_\omega \gamma_\mu \Psi A^{(\omega)\mu}$$

In the following lines we will omit the label ω , because we will consider only the terms which concern the ω meson. Using the metric tensor $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ we can rewrite the terms above as:

$$-\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma} + \frac{1}{2} m^2 A_\mu A_\rho \eta^{\mu\rho} - \Gamma \bar{\Psi} \gamma_\mu \Psi A_\rho \eta^{\mu\rho} \quad (3.10)$$

Using the definition of $F_{\mu\nu}$ we can write the first term as:

$$\begin{aligned} F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \eta^{\mu\rho} \eta^{\nu\sigma} \\ &= \partial_\mu A_\nu \partial_\rho A_\sigma \eta^{\mu\rho} \eta^{\nu\sigma} - \partial_\nu A_\mu \partial_\rho A_\sigma \eta^{\mu\rho} \eta^{\nu\sigma} \\ &\quad - \partial_\mu A_\nu \partial_\sigma A_\rho \eta^{\mu\rho} \eta^{\nu\sigma} + \partial_\nu A_\mu \partial_\sigma A_\rho \eta^{\mu\rho} \eta^{\nu\sigma} \end{aligned} \quad (3.11)$$

We can sum the first and fourth terms and the second and third, because of the symmetry of metric tensor.

We now want to calculate the variation of action δS . In order to do so, we divide the integral in two terms I_1 and I_2 as follows:

$$\delta S = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \int d^4x (-\partial_\mu \delta A_\nu \partial_\rho A_\sigma \eta^{\mu\rho} \eta^{\nu\sigma} + \partial_\mu \delta A_\nu \partial_\sigma A_\rho \eta^{\mu\rho} \eta^{\nu\sigma}) \\ I_2 &= \int d^4x (m^2 \delta A_\mu A_\sigma \eta^{\sigma\mu} + m^2 \delta A_\sigma A_\mu \eta^{\sigma\mu} - \bar{\Psi} \Gamma \gamma_\mu \Psi \delta A_\sigma \eta^{\sigma\mu}) \end{aligned}$$

We therefore find the following expression for the action S :

$$\delta S = \int d^4x \delta A_\nu (\partial_\mu F^{\mu\nu} + m^2 A^\mu - \Gamma \bar{\Psi} \gamma^\nu \Psi) = 0$$

which brings to the equation of motion for the ω meson:

$$\partial_\mu F^{(\omega)\mu\nu} + m_\omega^2 A^{(\omega)\nu} = \Gamma_\omega \bar{\Psi} \gamma^\nu \Psi \quad (3.12)$$

ρ meson By following the same procedure we find:

$$\partial_\mu \mathbf{F}^{(\rho)\mu\nu} + m_\rho^2 \mathbf{A}^{(\rho)\nu} = \Gamma_\rho \bar{\Psi} \gamma^\nu \tau_3 \Psi \quad (3.13)$$

σ meson We now consider the terms concerning the σ meson. The equation of motion for σ can be found by considering the following terms of the Lagrangian:

$$\frac{1}{2}(\partial_\mu \Phi_\sigma \partial^\mu \Phi_\sigma - m_\sigma^2 \Phi_\sigma^2) + \bar{\Psi} \Gamma_\sigma \Psi \Phi_\sigma \quad (3.14)$$

which we can write as:

$$\frac{1}{2}(\partial_\mu \Phi_\sigma \partial_\rho \Phi_\sigma \eta^{\mu\rho} - m_\sigma^2 \Phi_\sigma^2) + \bar{\Psi} \Gamma_\sigma \Psi \Phi_\sigma \quad (3.15)$$

As we did before for the ω meson, we calculate the variation of action δS , which has to be zero in order to find the equations of motion. We can write:

$$\delta S = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \int d^4x \frac{1}{2} (\partial_\mu \delta \Phi_\sigma \partial_\rho \Phi_\sigma \eta^{\mu\rho} + \partial_\mu \Phi_\sigma \partial_\rho \delta \Phi_\sigma \eta^{\mu\rho}) \\ I_2 &= \int d^4x (-m_\sigma^2 \Phi_\sigma \delta \Phi_\sigma + \Gamma_\sigma \bar{\Psi} \Psi \delta \Phi_\sigma) \end{aligned}$$

Hence, the equation of motion reads:

$$(\partial^\mu \partial_\mu + m_\sigma^2) \Phi_\sigma = \Gamma_\sigma \bar{\Psi} \Psi \quad (3.16)$$

δ meson A calculation similar to the one we made for the σ meson leads to the following equation of motion:

$$(\partial^\mu \partial_\mu + m_\delta^2) \Phi_\delta = \Gamma_\delta \bar{\Psi} \tau_3 \Psi \quad (3.17)$$

3.2 Mean-field equations

The mean-field approximation removes all quantum fluctuations of the meson fields and uses their expectation values. We will also work in the no-sea approximation, which consists in ignoring the anti-particle states. The Hartree mean field equations are obtained by calculating the ground state expectation value on both sides of the meson field equation of motion. We assume that the meson fields can be decomposed in a stationary part and in a time dependent part, the latter of which has a vanishing ground state expectation value:

$$\begin{aligned} \Phi_i(\mathbf{r}, t) &\longrightarrow \Phi_i(\mathbf{r}) + \delta\Phi_i(t) & i = \sigma, \delta \\ A^{(j)\mu}(\mathbf{r}, t) &\longrightarrow A^{(j)\mu}(\mathbf{r}) + \delta A^{(j)\mu}(t) & j = \omega, \rho \end{aligned}$$

We represent the nucleus ground state as $|0\rangle$, and we find:

$$\begin{aligned} \langle 0 | \Phi_i(\mathbf{r}, t) | 0 \rangle &\longrightarrow \Phi_i(\mathbf{r}) & i = \sigma, \delta \\ \langle 0 | A^{(j)\mu}(\mathbf{r}, t) | 0 \rangle &\longrightarrow A^{(j)\mu}(\mathbf{r}) & j = \omega, \rho \end{aligned}$$

Moreover, the time reversal symmetry implies that the vector potential space-like parts vanish: $A^{(j)\mu}(\mathbf{r}) \rightarrow A^{(j)0}(\mathbf{r})$.

From now on, we denote the static classical meson fields with Φ_i and $A^{(j)0}$.

Considering all the informations above and remembering that in the Lorentz gauge $\partial_\mu A^\mu = 0$, we can finally write the mean field equations:

$$(-\partial^i \partial_i + m_\sigma^2) \Phi_\sigma = \Gamma_\sigma \rho^s \quad (3.18)$$

$$(-\partial^i \partial_i + m_\omega^2) A^{(\omega)0} = \Gamma_\omega \rho \quad (3.19)$$

$$(-\partial^i \partial_i + m_\delta^2) \Phi_\delta = \Gamma_\delta \rho_3^s \quad (3.20)$$

$$(-\partial^i \partial_i + m_\rho^2) A^{(\rho)0} = \Gamma_\rho \rho_3 \quad (3.21)$$

where:

$$\rho^s \equiv \langle 0 | \bar{\Psi} \Psi | 0 \rangle = \rho_n^s + \rho_p^s \quad (3.22)$$

$$\rho \equiv \langle 0 | \bar{\Psi} \gamma_0 \Psi | 0 \rangle = \rho_n + \rho_s \quad (3.23)$$

$$\rho_3^s \equiv \langle 0 | \bar{\Psi} \tau_3 \Psi | 0 \rangle = \rho_n^s - \rho_p^s \quad (3.24)$$

$$\rho_3 \equiv \langle 0 | \bar{\Psi} \gamma_0 \tau_3 \Psi | 0 \rangle = \rho_n - \rho_s \quad (3.25)$$

We can also rewrite the Dirac equation as:

$$\left[\gamma_\mu \left(i\partial^\mu - \hat{\Sigma}_N^\mu \right) - (M - \Sigma_N^s) \right] \psi_N = 0 \quad N = n, p \quad (3.26)$$

The non zero contributions to the mean field self-energies are:

$$\Sigma_N^s = \Gamma_\sigma \Phi_\sigma + \tau_3 \Gamma_\delta \Phi_\delta$$

$$\Sigma_N^\mu = \Gamma_\omega A^{(\omega)\mu} + \Gamma_\rho \boldsymbol{\tau} A^{(\rho)\mu}$$

From the latter equation we find:

$$\Sigma_N^0 = \Gamma_\omega A^{(\omega)0} + \tau_N \Gamma_\rho A^{(\rho)0}$$

where $\tau_n = 1$ for neutrons and $\tau_p = -1$ for protons.

3.3 Infinite symmetric nuclear matter

In the infinite nuclear matter system under the relativistic Hartree description we can simplify the equations derived previously. In fact, we assume the system to be infinite, homogeneous and isotropic, thus, once fixed a certain density of nucleons, all the other quantities are constant and we have translational invariance. The solutions of the Dirac equation for neutrons and protons are the usual plane wave Dirac spinors.

The infinite matter equations for the meson fields are:

$$\Phi_\sigma = \frac{\Gamma_\sigma}{m_\sigma^2} \rho^s \quad (3.27)$$

$$A^{(\omega)0} = \frac{\Gamma_\omega}{m_\omega^2} \rho \quad (3.28)$$

$$\Phi_\delta = \frac{\Gamma_\delta}{m_\delta^2} \rho_3^s \quad (3.29)$$

$$A^{(\rho)0} = \frac{\Gamma_\rho}{m_\rho^2} \rho_3 \quad (3.30)$$

If we consider the case of infinite symmetric nuclear matter we have further simplifications: since the density of protons is equal to the one of neutrons, we have:

$$\rho_3^s = \rho_n^s - \rho_p^s = 0 \quad (3.31)$$

$$\rho_3 = \rho_n - \rho_p = 0 \quad (3.32)$$

As a consequence, the fields Φ_δ and $A^{(\rho)0}$ vanish and the system remains exclusively under the effect of the fields Φ_σ and $A^{(\omega)0}$. In particular:

$$\Sigma_N^s = \Gamma_\sigma \Phi_\sigma$$

$$\Sigma_N^0 = \Gamma_\omega A^{(\omega)0}$$

and

$$M^* = M - \Gamma_\sigma \Phi_\sigma$$

The nucleon and scalar densities can be found as follows:

$$\rho = \frac{4}{(2\pi)^3} \int_{|k| < k_F} d^3k = \frac{2k_F^3}{3\pi^2} \quad (3.33)$$

$$\rho^s = \frac{4}{(2\pi)^3} \int_{|k| < k_F} \frac{M^*}{E} d^3k = \frac{M^*}{\pi^2} \left[k_F E_F - M^{*2} \log \left(\frac{k_F + E_F}{M^*} \right) \right] \quad (3.34)$$

k_F is the Fermi momentum and E_F is the Fermi energy for nucleons in the

medium:

$$E_F = \sqrt{k_F^2 + M^{*2}}$$

We now calculate the energy and pressure density from the definition of the momentum-energy tensor $T^{\mu\nu}$, where:

$$T^{\mu\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi_i)} \partial_\nu \Phi_i - \eta^{\mu\nu} \mathcal{L} \quad (3.35)$$

and Φ_i runs for all possible fields.

We take the ground state expectation value of the components T^{00} (which represents the energy density e) and T^{ii} (which represents the pressure p) and we find:

$$e = \langle 0 | T^{00} | 0 \rangle = \frac{1}{4} [3E_F \rho + M^* \rho^s] + \frac{1}{2} [m_\sigma^2 \Phi_\sigma^2 + m_\omega^2 A^{(\omega)02}] \quad (3.36)$$

$$p = \frac{1}{3} \sum_{i=1}^3 \langle 0 | T^{ii} | 0 \rangle = \frac{1}{4} [3E_F \rho - M^* \rho^s] - \frac{1}{2} [m_\sigma^2 \Phi_\sigma^2 - m_\omega^2 A^{(\omega)02}] \quad (3.37)$$

From equation (3.36) one can calculate the binding energy per nucleon $\frac{E}{A}$ as:

$$\frac{E}{A} = \frac{e}{\rho} - M$$

We have also checked that the pressure as calculated thanks to the energy-momentum tensor is equal to the thermodynamical definition:

$$p = \rho^2 \frac{\partial \left(\frac{E}{A} \right)}{\partial \rho}$$

Chapter 4

Nonrelativistic reduction of the model

The aim of this chapter is to perform the non-relativistic reduction of the model we studied previously, in order to understand the differences between the relativistic and non-relativistic approaches for the description of the nuclear equation of state (i. e. the relation that occurs between the energy and the density of the system).

We will follow the procedure outlined by Reinhard in ref. [9].

We will try to understand if the results we obtain thanks to the reduction suffice to describe the symmetric matter equation of state and the relevance of the relativistic effects.

4.1 Derivation of an effective Hamiltonian

4.1.1 First order reduction

In order to simplify notation, we introduce the scalar and vector potentials [9]

$$\mathcal{S} \equiv -\Gamma_\sigma \Phi_\sigma \tag{4.1}$$

$$\mathcal{V} \equiv \Gamma_\omega A^{(\omega)0} \tag{4.2}$$

We will perform the expansion in orders of velocity $v = \frac{p}{M}$.

We want to derive an effective Hamiltonian up to order v^2 (we can derive the Hamiltonian from the Lagrangian thanks to the relation $H = T^{00}$). In order to do so, one can start by considering the energy expectation value before variation

$$\langle H - \epsilon \rangle = \int d^3r \begin{pmatrix} \psi^{(u)} \\ \psi^{(d)} \end{pmatrix}^\dagger \begin{pmatrix} M + \mathcal{S} + \mathcal{V} - \epsilon & \boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -M + \mathcal{S} + \mathcal{V} - \epsilon \end{pmatrix} \begin{pmatrix} \psi^{(u)} \\ \psi^{(d)} \end{pmatrix} \quad (4.3)$$

where

$$\Psi = \begin{pmatrix} \psi^{(u)} \\ \psi^{(d)} \end{pmatrix}$$

is the nucleon wavefunction (Dirac spinor) and $\psi^{(u)}$ and $\psi^{(d)}$ are respectively the upper and lower components of Ψ .

A first simplification consists in writing the lower component as a function of the upper, according to the Dirac equation:

$$\psi^{(d)} = \mathcal{B} \boldsymbol{\sigma} \cdot \mathbf{p} \psi^{(u)} \quad (4.4)$$

where

$$\mathcal{B} = \frac{1}{M + \epsilon + \mathcal{S} + \mathcal{V}} \quad (4.5)$$

By calculating the products inside the integral in equation (4.3) and thanks to equation (4.4) we find the expression:

$$\langle H - \epsilon \rangle = \int d^3r \psi^{(u)\dagger} (M + \mathcal{S} + \mathcal{V} - \epsilon + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B} \boldsymbol{\sigma} \cdot \mathbf{p}) \psi^{(u)} \quad (4.6)$$

where the term $\boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B} \boldsymbol{\sigma} \cdot \mathbf{p}$ represents the kinetic term of the equation, which is at least of the second order. We expand $\mathcal{B}$ up to the second order with respect to $\frac{\mathcal{S} + \mathcal{V} - \epsilon'}{2(\mathcal{S} + \mathcal{V})}$ as follows:

$$\mathcal{B} = \frac{1}{2(M + \mathcal{S}) + (\epsilon' - (\mathcal{S} + \mathcal{V}))} = \mathcal{B}_0 + \mathcal{B}_0^2(\mathcal{S} + \mathcal{V} - \epsilon') + \mathcal{O}(\mathcal{B}_0^3) \quad (4.7)$$

where

$$\mathcal{B}_0 = \frac{1}{2(M + \mathcal{S})} \quad (4.8)$$

is the inverse of the Dirac effective mass M^* multiplied by $\frac{1}{2}$ and $\epsilon' = \epsilon - M$.

Using this expansion we can rewrite equation (4.6) :

$$\begin{aligned} \langle H - \epsilon \rangle = \int d^3r \psi^{(u)\dagger} (\mathcal{S} + \mathcal{V} + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0 \boldsymbol{\sigma} \cdot \mathbf{p}) \psi^{(u)} \\ + \psi^{(u)\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 (\mathcal{S} + \mathcal{V}) \boldsymbol{\sigma} \cdot \mathbf{p} - \epsilon' (1 + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \mathbf{p})) \psi^{(u)} \end{aligned} \quad (4.9)$$

The energy ϵ' appears together with the norm

$$Norm = \int d^3r \psi^{(u)\dagger} (1 + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \mathbf{p}) \psi^{(u)}$$

The renormalization with the norm kernel

$$\hat{I} = 1 + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \mathbf{p} \quad (4.10)$$

shows that $\psi^{(u)}$ is not directly a probability amplitude. We look for an equivalent classical wavefunction $\psi^{(c)}$ with the standard probability interpretation and normalisation. Therefore we introduce the transformation

$$\begin{aligned} \psi^{(c)} &= \hat{I}^{\frac{1}{2}} \psi^{(u)} \\ \psi^{(u)} &= \hat{I}^{-\frac{1}{2}} \psi^{(c)} \end{aligned} \quad (4.11)$$

Thanks to this transformation we can write the energy expectation value as

$$\langle H - \epsilon \rangle = \int d^3r \psi^{(c)\dagger} (H_{eff}^{(class)} - \epsilon') \psi^{(c)} \quad (4.12)$$

The effective classical Hamiltonian is given by the transformation

$$H_{eff}^{(class)} = \hat{I}^{-\frac{1}{2}} [\mathcal{S} + \mathcal{V} + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0 \boldsymbol{\sigma} \cdot \mathbf{p} + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 (\mathcal{S} + \mathcal{V}) \boldsymbol{\sigma} \cdot \mathbf{p}] \hat{I}^{-\frac{1}{2}} \quad (4.13)$$

To calculate $H_{eff}^{(class)}$ we can expand the operator

$$\hat{I}^{-\frac{1}{2}} = 1 - \frac{1}{2} \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \mathbf{p}$$

By inserting it into equation (4.13) and keeping all terms up to order v^2 , we can write the effective classical Hamiltonian as:

$$H_{eff}^{(class)} = \mathcal{S} + \mathcal{V} + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0 \boldsymbol{\sigma} \cdot \mathbf{p} + \mathcal{O}(v^4) \quad (4.14)$$

We now rewrite the equation for the case of spherical symmetry. That means all the fields are radially symmetric, and we can use the relation:

$$(\boldsymbol{\sigma} \cdot \mathbf{p})(\boldsymbol{\sigma} \cdot \mathbf{p}) = \mathbf{p} \cdot \mathbf{p} + i\boldsymbol{\sigma}(\mathbf{p} \times \mathbf{p})$$

We therefore find:

$$H_{eff}^{(class)} = \mathcal{S} + \mathcal{V} + \mathbf{p} \mathcal{B}_0 \mathbf{p} \quad (4.15)$$

We can rewrite this equation as:

$$H_{eff}^{(class)} = -\Gamma_\sigma \Phi_\sigma + \Gamma_\omega A^{(\omega)0} + \frac{p^2}{2M^*} \quad (4.16)$$

where

$$T = \frac{p^2}{2M^*}$$

is the kinetic term and

$$U = -\Gamma_\sigma \Phi_\sigma + \Gamma_\omega A^{(\omega)0}$$

is the potential.

From this expression is evident that the non-relativistic effective mas is different than the relativistic one.

4.1.2 Second order reduction

Our aim here is to expand the non-relativistic reduction up to order v^4 , in order to study its effects on the results we will present in the next chapter.

We consider again equation (4.13) and we perform the expansion by keeping all terms up to order v^4 :

$$\begin{aligned} H_{eff}^{(class)} = & \mathcal{S} + \mathcal{V} + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0 \boldsymbol{\sigma} \cdot \mathbf{p} + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 (\mathcal{S} + \mathcal{V}) \boldsymbol{\sigma} \cdot \mathbf{p} \\ & - \frac{1}{2} \{ \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \mathbf{p}, \mathcal{S} + \mathcal{V} + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0 \boldsymbol{\sigma} \cdot \mathbf{p} \} \end{aligned} \quad (4.17)$$

We would like to write the fourth-order term in a more symmetric form. To do so, we use $(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})^2 = \nabla^2$:

$$\frac{1}{2} \{ \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \mathbf{p}, \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0 \boldsymbol{\sigma} \cdot \mathbf{p} \} = (\boldsymbol{\sigma} \cdot \mathbf{p})^2 \mathcal{B}_0^3 (\boldsymbol{\sigma} \cdot \mathbf{p})^2 + \boldsymbol{\sigma} \cdot \mathbf{p} [\mathcal{B}_0 (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \mathcal{B}_0)^2 - \frac{1}{2} \mathcal{B}_0^2 \nabla^2 \mathcal{B}_0] \boldsymbol{\sigma} \cdot \mathbf{p}$$

and

$$\frac{1}{2} \{ \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 \boldsymbol{\sigma} \cdot \mathbf{p}, \mathcal{S} + \mathcal{V} \} = \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0^2 (\mathcal{S} + \mathcal{V}) \boldsymbol{\sigma} \cdot \mathbf{p} - \mathcal{B}_0 (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \mathcal{B}_0) [\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} (\mathcal{S} + \mathcal{V})] - \frac{1}{2} \mathcal{B}_0^2 \nabla^2 (\mathcal{S} + \mathcal{V})$$

where we have used the property $\{AB, C\} = \{A, C\}B + A\{B, C\}$.

We can thus rewrite the effective Hamiltonian as follows:

$$\begin{aligned}
H_{eff}^{(class)} = & \mathcal{S} + \mathcal{V} + \boldsymbol{\sigma} \cdot \mathbf{p} \mathcal{B}_0 \boldsymbol{\sigma} \cdot \mathbf{p} + \mathcal{B}_0 (\boldsymbol{\sigma} \cdot \nabla \mathcal{B}_0) [\boldsymbol{\sigma} \cdot \nabla (\mathcal{S} + \mathcal{V})] + \frac{1}{2} \mathcal{B}_0^2 \nabla^2 (\mathcal{S} + \mathcal{V}) \\
& - \boldsymbol{\sigma} \cdot \mathbf{p} [\mathcal{B}_0 (\boldsymbol{\sigma} \cdot \nabla \mathcal{B}_0)^2 - \frac{1}{2} \mathcal{B}_0^2 (\nabla^2 \mathcal{B}_0)] \boldsymbol{\sigma} \cdot \mathbf{p} + (\boldsymbol{\sigma} \cdot \mathbf{p})^2 \mathcal{B}_0 (\boldsymbol{\sigma} \cdot \mathbf{p})^2
\end{aligned} \tag{4.18}$$

We now rewrite the Hamiltonian in case of spherical symmetry. The fields can be written as

$$\nabla \mathcal{F} = \frac{\mathbf{r}}{r} \frac{d}{dr} \mathcal{F} \quad \mathcal{F} = \mathcal{S}, \mathcal{V}, \mathcal{B}$$

and we use:

$$\boldsymbol{\sigma} \cdot \mathbf{r} \boldsymbol{\sigma} \cdot \mathbf{p} = \mathbf{r} \cdot \mathbf{p} + i \boldsymbol{\sigma} \cdot (\mathbf{r} \times \mathbf{p}) = -ir \frac{d}{dr} + i \boldsymbol{\sigma} \cdot \mathbf{L}$$

where $\mathbf{L}$ represents the orbital angular momentum.

We can thus write the classical effective Hamiltonian in case of spherical symmetry:

$$H_{eff}^{(class)} = \mathcal{S} + \mathcal{V} + \frac{1}{2} \frac{d}{dr} \left(\mathcal{B}_0^2 \frac{d}{dr} (\mathcal{S} + \mathcal{V}) \right) + \mathbf{p} \cdot \tilde{\mathcal{B}} \mathbf{p} + \left(\frac{1}{r} \frac{d}{dr} \tilde{\mathcal{B}} \right) \boldsymbol{\sigma} \cdot \mathbf{L} + \mathbf{p}^2 \mathcal{B}_0^3 \mathbf{p}^2 \tag{4.19}$$

where

$$\tilde{\mathcal{B}} = \mathcal{B}_0 - \frac{1}{2} \mathcal{B}_0^2 \Delta \mathcal{B}_0 + \mathcal{B}_0 \left(\frac{d}{dr} \mathcal{B}_0 \right)^2$$

We now ignore the spin-orbit term and the other terms that depends from the derivatives with respect to r , because we are studying a system which is homogeneous and isotropic.

We can therefore write the effective Hamiltonian for infinite symmetric nuclear matter up to order v^4 as:

$$H_{eff}^{(class)} = \mathcal{S} + \mathcal{V} + \mathbf{p} \mathcal{B}_0 \mathbf{p} - \mathbf{p} \mathcal{B}_0^2 (\mathcal{S} + \mathcal{V}) \mathbf{p} + \mathbf{p}^2 \mathcal{B}_0^3 \mathbf{p}^2 \tag{4.20}$$

which is equal to:

$$H_{eff}^{(class)} = \mathcal{S} + \mathcal{V} + \frac{p^2}{2M^*} + \frac{p^4}{(2M^*)^3} \quad (4.21)$$

4.2 Infinite symmetric nuclear matter

Now that we have derived following Ref. [9] the Hamiltonian of the system up to orders v^2 and v^4 , we can calculate the binding energy for symmetric nuclear matter in the non relativistic approximation and compare it with the relativistic results.

The non-relativistic interaction is a zero-range density dependent interaction. The mesons we used to describe the relativistic model have large masses with respect to the pion mass (the σ meson, for example, has a mass of 550 MeV [12] and it is the lighter meson considered here), thus this approximation is reasonable.

4.2.1 First order reduction

The Hamiltonian can be written as:

$$H = T + U$$

where $T = \frac{p^2}{2M^*}$ is the kinetic term and $U = \mathcal{S} + \mathcal{V}$ is the potential term.

Thus, the binding energy per nucleon $\frac{E}{A}$ can be found as

$$\frac{E}{A} = \frac{T}{A} + \frac{U}{A} \quad (4.22)$$

We know that the density ρ can be calculated in momentum space as

$$\rho = \frac{\nu}{(2\pi)^3} \int_{|k| \leq k_F} d^3k = \frac{2k_F^3}{3\pi^2} \quad (4.23)$$

where ν stands for the spin and isospin degeneracy: $\nu = \nu_S \cdot \nu_T = 2 \cdot 2$, where ν_S stands for the spin degeneracy (which is equal to 2, in fact we have two possible spin states: up and down) and ν_T stands for the isospin degeneracy

(also equal to 2 because we have two possible isospin states: neutron and proton).

Thanks to this expression one can find the Fermi momentum for nucleons as a function of the baryon density:

$$k_F = \left(\frac{3\pi}{2} \rho \right)^{\frac{1}{3}} \quad (4.24)$$

We now calculate the kinetic and potential energy for nucleons in the Hartree approximation, in order to find the binding energy of nucleons for the nonrelativistic limit according to equation (4.22).

Kinetic energy We calculate this term by integrating the expression of the kinetic term T of the Hamiltonian in the momentum space for $|k| \leq k_F$:

$$\begin{aligned} \frac{T}{A} &= \frac{\frac{\nu}{(2\pi)^3} \int_{|k| \leq k_F} d^3k \frac{k^2}{2M^*}}{\frac{\nu}{(2\pi)^3} \int_{|k| \leq k_F} d^3k} \\ &= \frac{1}{2M^*} \frac{\int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^{k_F} dk \, k^2 \cdot k^2 \sin \theta}{\int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^{k_F} dk \, k^2 \sin \theta} \\ &= \frac{1}{\pi^2 M^*} \frac{\frac{k_F^5}{5}}{\frac{k_F^3}{3}} \end{aligned} \quad (4.25)$$

We thus obtain the following expression:

$$\frac{T}{A} = \frac{3}{5} \frac{k_F^2}{2M^*} \quad (4.26)$$

Potential energy Following the same procedure we used for the kinetic energy, we can find the potential contribute to the binding energy per nucleon as:

$$\begin{aligned}\frac{U}{A} &= \frac{\frac{\nu}{(2\pi)^3} \int_{|k| \leq k_F} d^3k (\mathcal{S} + \mathcal{V})}{\frac{\nu}{(2\pi)^3} \int_{|k| \leq k_F} d^3k} \\ &= (\mathcal{S} + \mathcal{V})\end{aligned}\tag{4.27}$$

Therefore, the expression for the potential energy is:

$$\frac{U}{A} = (\mathcal{S} + \mathcal{V})\tag{4.28}$$

where we have considered that in the infinite nuclear matter approximation $\mathcal{S}$ and $\mathcal{V}$ can only depend on the density and we have assumed their value is the one obtained from the relativistic calculations performed in the next chapter.

Binding energy We can thus calculate the binding energy per nucleon according to equation (4.22).

$$\frac{E}{A} = \mathcal{S} + \mathcal{V} + \frac{3}{5} \frac{k_F^2}{2M^*}\tag{4.29}$$

4.2.2 Second order reduction

We consider now the Hamiltonian in equation (5.16), and rewrite it as:

$$H = T + U + T_2$$

where

$$T_2 = \frac{p^4}{(2M^*)^3}$$

We can thus calculate the new correction to the energy:

$$\begin{aligned}\frac{T_2}{A} &= \frac{\frac{\nu}{(2\pi)^3} \int_{|k| \leq k_F} d^3k \frac{p^4}{(2M^*)^3}}{\frac{\nu}{(2\pi)^3} \int_{|k| \leq k_F} d^3k} \\ &= \frac{3}{7} \frac{k_F^4}{(2M^*)^3}\end{aligned}\tag{4.30}$$

We can thus write the binding energy per nucleon up to order v^4 :

$$\frac{E}{A} = \mathcal{S} + \mathcal{V} + \frac{3}{5} \frac{k_F^2}{2M^*} + \frac{3}{7} \frac{k_F^4}{(2M^*)^3} \quad (4.31)$$

We notice that the binding energy does not depend explicitly on the density of the system. The density dependence can be seen if one considers the expressions for $\mathcal{S}, \mathcal{V}, k_F$ and M^* , which depend on the value of ρ .

Also the meson fields give a contribution to the energy, thanks to the scalar and vector potentials but also thanks to the effective mass, which depends on the σ meson.

Chapter 5

Results

We calculated the binding energy per nucleon of symmetric nuclear matter in both the relativistic and non-relativistic approach, together with some other quantities related such as the pressure and the incompressibility.

We'll also present some figures in order to discuss the differences between the two models.

In this section we'll use the following values for the coupling constants, meson masses, density and energy:

- Baryonic density for the saturation point of symmetric nuclear matter
 $\rho = 0.16 \text{ fm}^{-3}$
- Saturation energy of symmetric nuclear matter: $\frac{E}{A} = -16 \text{ MeV}$
- σ meson mass: $m_\sigma = 550 \text{ MeV}$ ([12])
- ω meson mass: $m_\omega = 780 \text{ MeV}$ ([12])
- nucleons mass: $M = 939 \text{ MeV}$ ([12])

The values of the couplings constants have been fitted to reproduce the saturation density and energy of symmetric nuclear matter. We thus found $\Gamma_\sigma = 10.653$ and $\Gamma_\omega = 13.143$. We chose to take the physical values for the masses, even if the model we studied does not allow us to relate to QCD. The σ meson mass, instead, has been evaluated by fitting experimental data

(this meson, in fact, is not real and it can be considered as two correlated pions).

Finally, we can explain the value we fixed for the energy thanks to the Bethe-Weizsäcker formula [13].

$$\frac{E}{A}(Z, A) = a_1 A + a_2 A^2 + a_3 \frac{Z^2}{A} + a_4 \frac{(N-2)^2}{A} \pm a_5 A^{\frac{-3}{4}} \quad (5.1)$$

where A represents the mass number of the nucleus, Z its atomic number and N the neutron number. The quantities a_i , $i = 1, \dots, 5$, moreover, represents coupling constants of the system. The first term of this semi-empirical formula is a volume term which describes the energy given by the short range interaction between close nucleons. This energy represents the saturation binding energy we are considering. The constant a_1 is equal for protons and neutrons and takes the value $a_1 = 16$ MeV. By inserting this value into the first term of equation (5.1), one finds approximately the value for the saturation density we fixed.

We begin by describing the results we found thanks to the relativistic model, and then we will discuss to the non-relativistic reduction.

5.1 Relativistic hadron field theory

Scalar density In order to calculate the binding energy per nucleon we need to find the scalar density ρ_s of the system, whose expression is given by:

$$\rho^s = \frac{4}{(2\pi)^3} \int_{|k| < k_F} \frac{M^*}{E} d^3k = \frac{M^*}{\pi^2} \left[k_F E_F - M^{*2} \log \left(\frac{k_F + E_F}{M^*} \right) \right] \quad (5.2)$$

The equation can not be solved analytically because of the dependence of both k_F and M^* (and consequently of E_F) from the scalar density. We therefore decided to solve it numerically thanks to a recursive method.

We fixed ρ and introduced a field Φ_0 such that

$$\Phi_0 = \left(\frac{\Gamma_\sigma}{m_\sigma} \right)^2 \rho (\hbar c)^3 \quad (5.3)$$

where $(\hbar c)^3$ is used to have the field value expressed in MeV.

At the first step of the iteration ($i = 1$), for a fixed ρ , we gave the field $\Phi_\sigma(i = 1)$ the value of Φ_0 . We used this field to calculate the effective mass M^* and consequently the Fermi energy E_F of nucleons. Then, we calculated the new value for the scalar density ρ^s thanks to equation (5.2), and we used this density value to calculate the new value for the field Φ_σ . We iterate this procedure until the condition

$$|\Phi_\sigma(i) - \Phi_\sigma(i - 1)| < 10^{-8} \quad (5.4)$$

was verified.

The value of ρ^s that satisfies equation (5.4) is the the solution for the scalar density within the required accuracy.

For $\rho = 0.16 \text{ fm}^{-3}$ we obtained $\rho^s = 0.148791 \text{ fm}^{-3}$.

Then, in order to construct the figures of other quantities such as the effective mass, we decided to find the values of ρ^s for values of ρ varying from 0 to 0.3 fm^{-3} . The results are presented in figure 5.1 We chose this range for the density for two reasons. If we look at values of the density $\rho > 0.3 \text{ fm}^{-3}$, in fact, we notice that one might see other degrees of freedom (such as hyperions, quarks...) because we are studying a situation in which the density is so large that nucleons start to move closer: the inter nucleon distance is of the order of 0.8 fm^{-3} , which is of the order of the radius of nucleons (for example the radius of the proton is $r_p \sim 0.8 \text{ fm}$ [14]). On the other hand, for ρ smaller than the saturation density the symmetric nuclear matter is unstable (cf. Figure 5.5) and forms nuclei. If we look inside of nuclei, however, we see that the interaction is short range, this means that the interaction range is smaller than the typical size of a heavy nucleus, which is $R_N = r_0 A^{\frac{1}{3}}$, where $r_0 = 1.25 \text{ fm}$ [2]. Therefore, nucleons which are inside these nuclei can only interact with the closer nucleons and not with the ones close to the surface, which are at a distance which is bigger

than the interaction range. Thus, one can expect the inside of the nucleus to behave as an infinite matter system, so we can consider also densities below the saturation one.

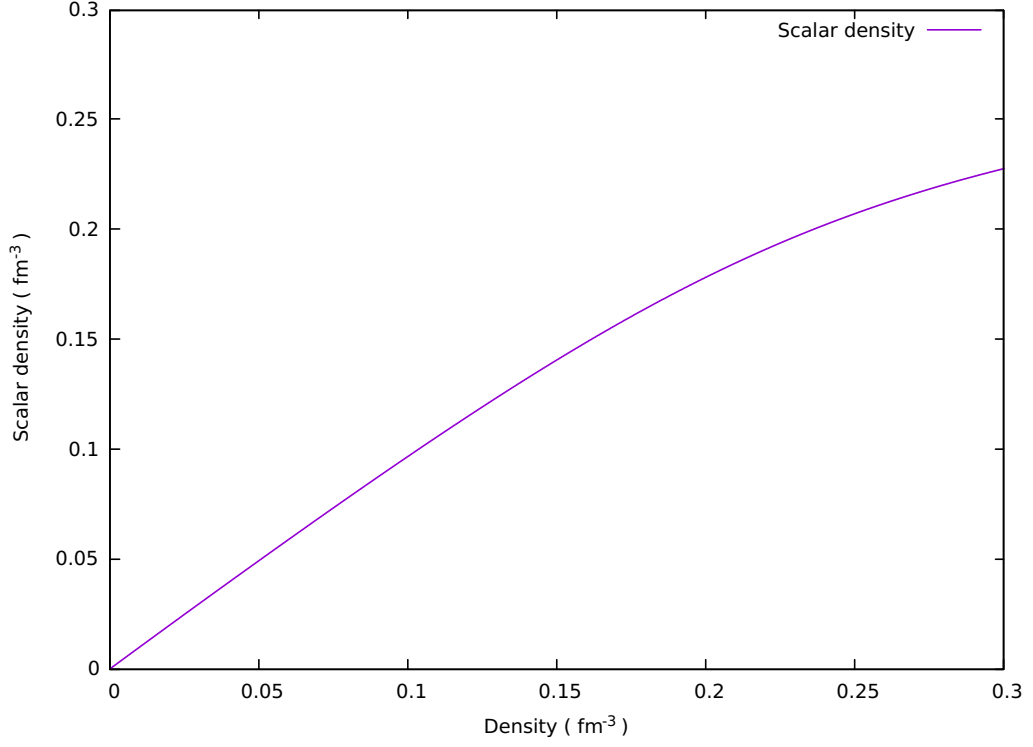


Figure 5.1: ρ^s as a function of ρ

From this Fig. 5.1 we can see that ρ^s is not exactly equal to ρ : the scalar density values are lower than the density ones. To explain this, we consider the equation necessary to find the scalar density and the baryon density:

$$\rho^s = \frac{4}{(2\pi)^3} \int_{|k| < k_F} \frac{M^*}{E} d^3k \quad \rho = \frac{4}{(2\pi)^3} \int_{|k| < k_F} d^3k \quad (5.5)$$

As we notice, the integrals differ for a multiplicative factor, which represents the mass-to-energy ratio. If we consider the expression for the energy

$$E = \sqrt{k^2 + M^{*2}}$$

we notice that the effective mass is always smaller than the energy.

The scalar density, thus, is smaller than the baryon density by the integral of this ratio.

Also, the curve is not linear, as we can see if we look at the values of ρ^s for $\rho > 0.15 \text{ fm}^{-3}$.

Effective mass Thanks to the scalar density, one can now look at the effective mass trend for values of the baryon density from 0 to 0.3 fm^{-3} .

We recall the definition of the relativistic effective mass:

$$M^* = M - \Gamma_\sigma \Phi_\sigma \quad (5.6)$$

where M is the nucleon mass and Γ_σ and Φ_σ are respectively the coupling constant and the field associated with the σ meson.

We can write this equation as:

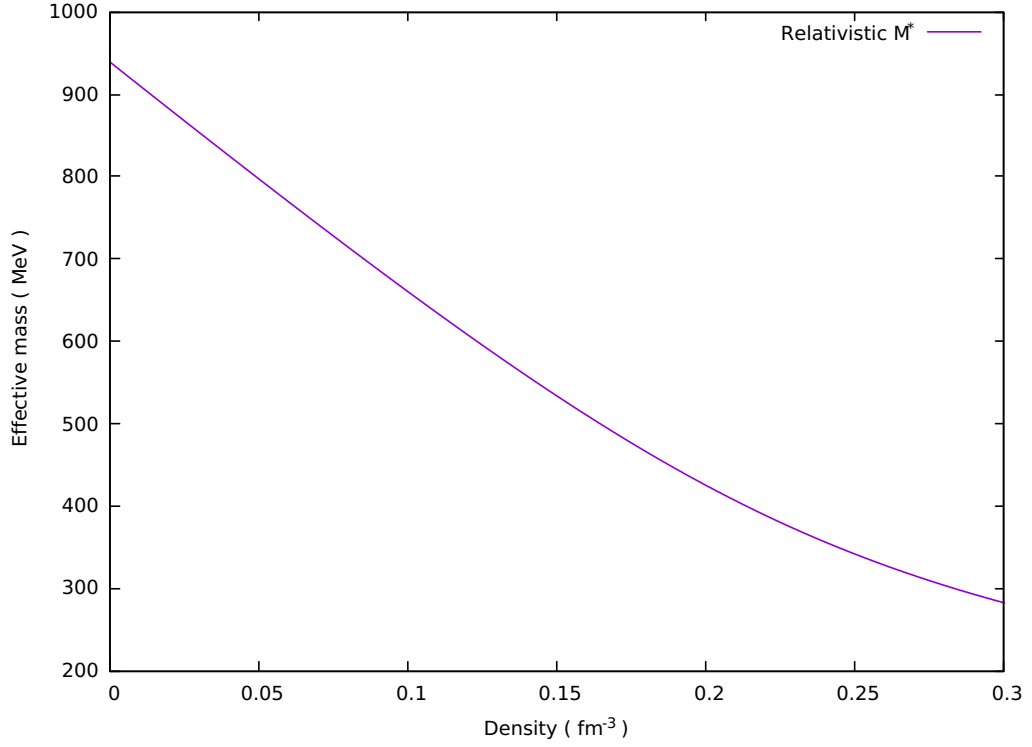
$$M^* = M - \left(\frac{\Gamma_\sigma}{m_\sigma} \right)^2 \rho^s \quad (5.7)$$

which shows the dependence of the effective mass from the scalar density ρ^s .

First of all, we calculated the effective mass of the system for fixed $\rho = 0.16 \text{ fm}^{-3}$, and we obtained $M^* = 510.102 \text{ MeV}$. For $\rho = 0 \text{ fm}^{-3}$ the effective mass is equal to the nucleon mass $M = 939 \text{ MeV}$, which is a strong reduction.

By varying the values of ρ we obtain a graph for the effective mass, shown in figure 5.2. We notice that the effective mass is a decrescent function of the density. As ρ increases, in fact, the σ field increases, and the effective mass decreases as shown in equation (5.7). We also notice that in the range of values we choose for the baryonic density, the effective mass is always positive.

One can find negative values for the effective mass for values of ρ which are approximately three times the value of the saturation density ($\rho = 0.16 \text{ fm}^{-3}$). This results in an inter-nucleon distance of about 0.8 fm , which is smaller than the radius of a proton, which means that a nucleon starts to see quark degrees of freedom because the nucleons move closer on average. This model, however, does not consider quarks degrees of freedom and therefore

Figure 5.2: M^* as a function of ρ

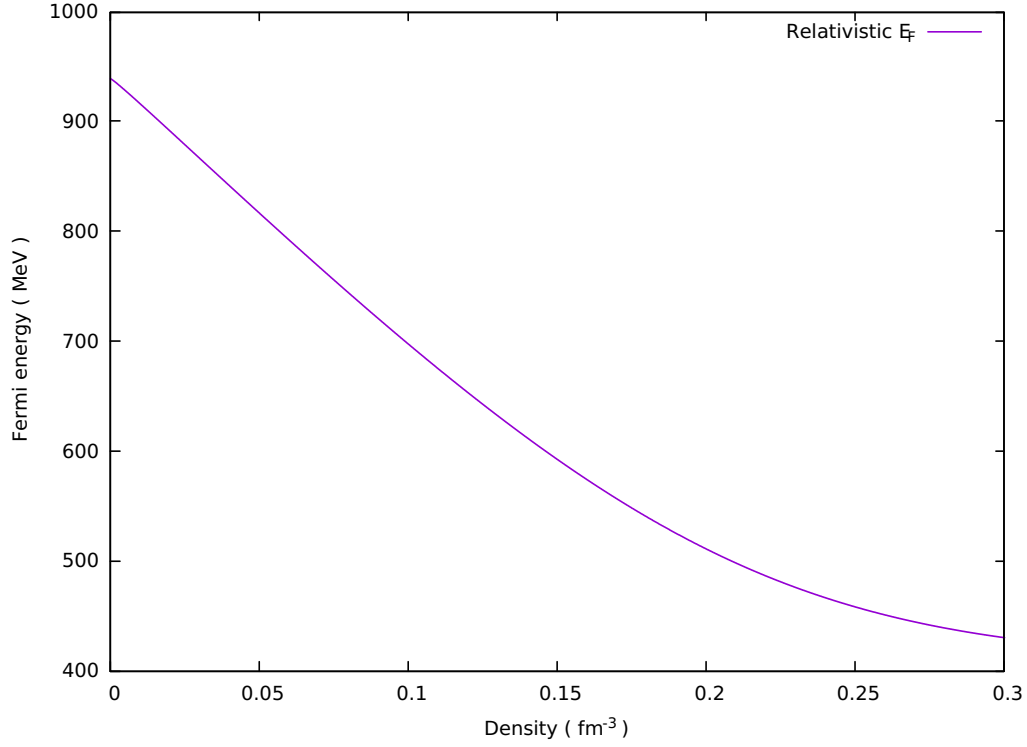
it cannot be applied to the study of very large densities.

Fermi Energy The dependence of the Fermi energy by the density is not explicit in its expression:

$$E_F = \sqrt{(M^*)^2 + k_F^2} \quad (5.8)$$

However, the effective mass and the Fermi momentum depend from ρ , and this results in a consequent dependence of the Fermi Energy E_F from the baryonic density. For $\rho = 0.16 \text{ fm}^{-3}$, we obtain $E_F = 573.929 \text{ MeV}$. The results for other densities are shown in figure 5.3.

We notice that also the Fermi energy is decreasing with the density, and that it's always positive. These facts are consequences of the dependence of the effective mass by the density and of the analytical expression of E_F .

Figure 5.3: E_F as a function of ρ

Energy density Knowing the scalar density, the effective mass and the Fermi energy, we are now able to calculate the energy density e and the pressure p according to the equations we found thanks to the energy momentum tensor $T^{\mu\nu}$.

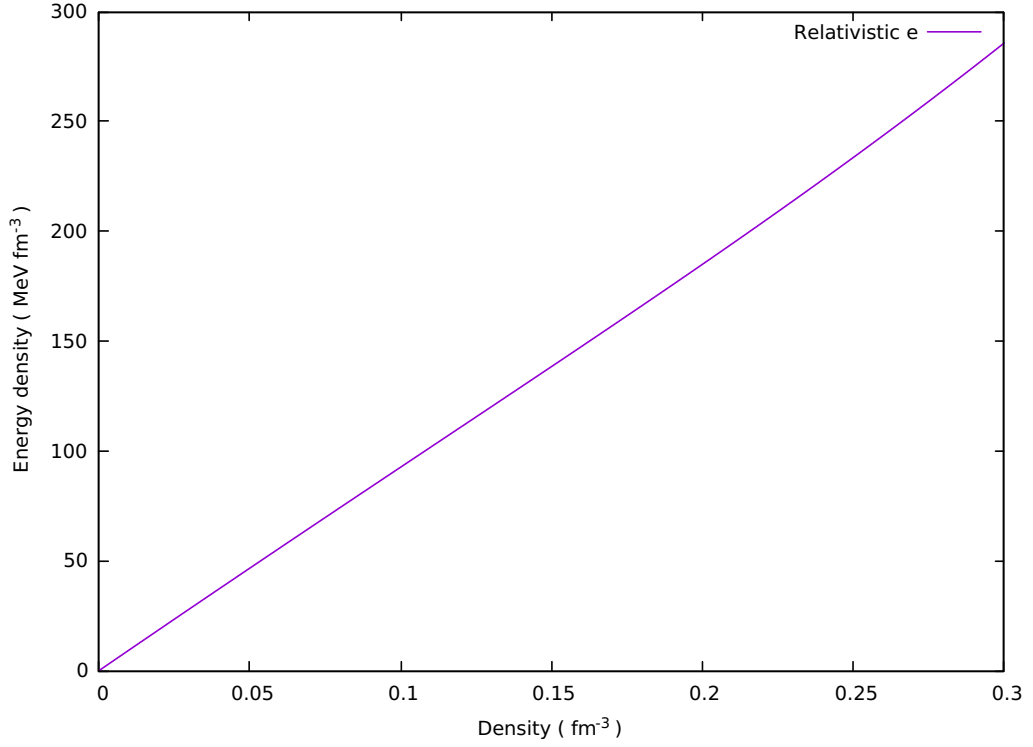
We recall the results we found in Chapter 3 and rewrite the expression for the energy density:

$$e = \langle 0 | T^{00} | 0 \rangle = \frac{1}{4} [3E_F \rho + M^* \rho^s] + \frac{1}{2} [m_\sigma^2 \Phi_\sigma^2 + m_\omega^2 A^{(\omega)02}] \quad (5.9)$$

We fixed $\rho = 0.16 \text{ fm}^{-3}$ and we found $e = 147.678 \text{ MeVfm}^{-3}$.

For ρ varying from 0 to 0.3 fm^{-3} we obtain a curve shown in figure 5.4

We notice that e is a crescent function of the density and that is non negative. The latter fact can be explained if we look at the analytical ex-

Figure 5.4: e as a function of ρ

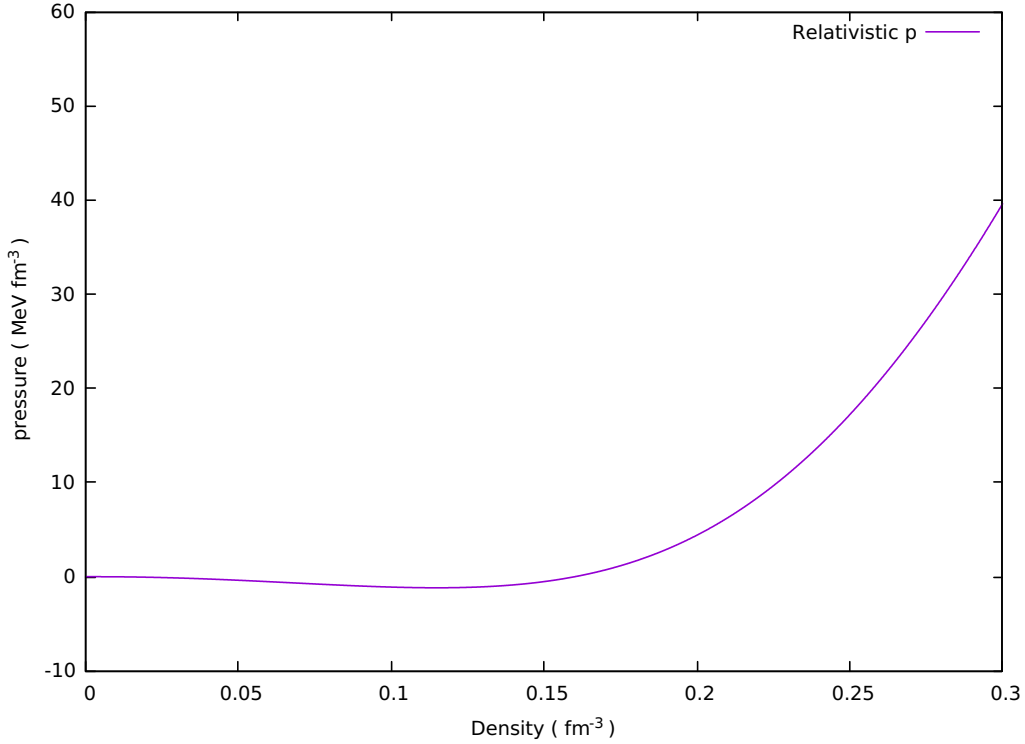
pression of e (equation (5.9)): the energy, in fact, density is given by the sum of non-negative terms.

Pressure As we did for the energy density, we recall the expression for the pressure given by the energy-momentum tensor.

$$p = \frac{1}{3} \sum_{i=1}^3 \langle 0 | T^{ii} | 0 \rangle = \frac{1}{4} [3E_F \rho - M^* \rho^s] - \frac{1}{2} [m_\sigma^2 \Phi_\sigma^2 - m_\omega^2 A^{(\omega)02}] \quad (5.10)$$

As expected, for $\rho = 0.16 \text{ fm}^{-3}$ we find $p = -0.002 \text{ MeV fm}^{-3}$. For this value of the density, the pressure should be zero. In this case we obtain a value which is not exactly zero due to numerical accuracy. The results for other densities are shown in figure 5.5.

We notice that for values of the density lower than the saturation density

Figure 5.5: p as a function of ρ

the pressure is negative. This means that we can study the model of symmetric nuclear matter for values of the density which are equal to the saturation density or higher, while if we look at the lower ones we see that symmetric matter is unstable and form nuclei. However, as we discussed previously, one may still learn from the energy per particle of symmetric nuclear matter at some sub-saturation densities since it might be suitable for the description of the bulk part of nuclei.

We checked that the pressure derived from the energy momentum tensor is equal to the one given by the thermodynamical definition:

$$p = \rho^2 \frac{\partial(\frac{E}{A})}{\partial \rho}$$

where $\frac{E}{A}$ is the binding energy per nucleon. We found that the two pressures are equal within 10^{-4} . This is a very strong test of the theory: the fact that

the pressures are equal within this uncertainty tells us that our theory is thermodynamically consistent.

Binding energy per nucleon At the end of this section we show the figure for the binding energy per nucleon of our relativistic model. First of all, we recall the expression for the binding energy per particle:

$$\frac{E}{A} = \frac{e}{\rho} - M \quad (5.11)$$

We imposed the value of $\frac{E}{A}$ for $\rho = 0,16 \text{ fm}^{-3}$ to be $\frac{E}{A} = -16 \text{ MeV}$ and to correspond to the minimum, as suggested by experimental data.

The graph is presented in figure 5.6, and it shows the typical trend of the binding energy per nucleon in a system described by our model (see, for example, [15]).

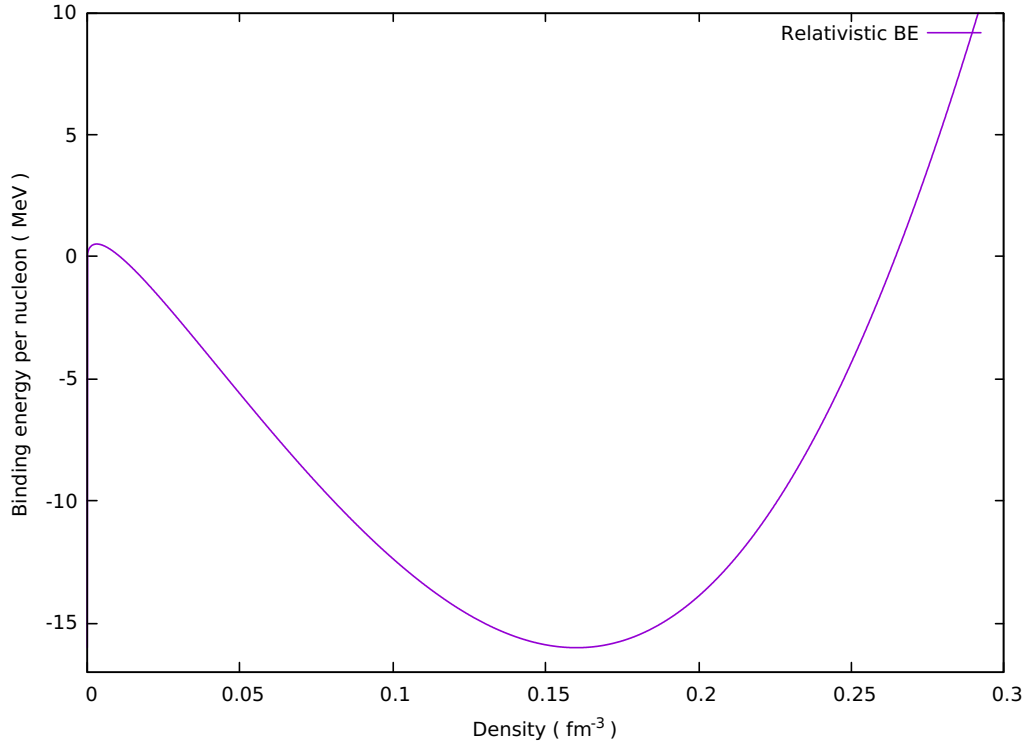


Figure 5.6: $\frac{E}{A}$ as a function of ρ

If we look at the values of the binding energy for $\rho < 0.16 \text{ fm}^{-3}$, we notice that we have negative energy states corresponding to negative values of the pressure. Again, we can explain this by saying that under the saturation density symmetric matter is unstable and forms nuclei.

From the expression of the binding energy we can also find the incompressibility K which is defined at $\rho = 0.16 \text{ fm}^{-3}$:

$$K = 9\rho^2 \frac{\partial^2(\frac{E}{A})}{\partial \rho^2}$$

and we find $K = 585.058 \text{ MeV}$, which is far from the one currently accepted for this quantity (around 230 MeV , Ref. [16]). This can be due to the approximations we made for the model: the density dependence of the potentials is not realistic enough to fit simultaneously density, energy and incompressibility at the saturation point.

5.2 Non relativistic reduction of the model

We now present the results we found for the non-relativistic reduction of our model. In the previous Chapter we described how to derive an effective Hamiltonian and how to calculate the binding energy of the system. Once known the binding energy, we also found the expressions for the energy density and the thermodynamical pressure of the non-relativistic model.

We used the values found before for the scalar density, the relativistic effective mass, the Fermi momentum and the Fermi energy of the system as functions of the density, so we won't present these results again.

In this section we'll discuss the results of the non relativistic binding energy, energy density and pressure.

5.2.1 First order reduction

Binding energy Given the Hamiltonian H of the system,

$$H = \frac{p^2}{2M^*} + U$$

one can find the expression of the binding energy per nucleon as:

$$\frac{E}{A} = \frac{1}{\rho} \frac{E}{V} = \mathcal{S} + \mathcal{V} + \frac{3}{5} \frac{k_F^2}{2M^*} \quad (5.12)$$

For $\rho = 0.16 \text{ fm}^{-3}$ we obtain $\frac{E}{A} = -54.0807 \text{ MeV}$. This value for the energy is very different from the one we fixed in the relativistic framework. This fact can find an explanation in the simplifications we made to derive the non-relativistic model. For example, we kept only the terms up to order v^2 while finding the effective Hamiltonian, which is a strong reduction that can explain the value we find for the first order non-relativistic binding energy. Looking at the graph in figure 5.7 one can notice the minimum of the energy for a value of the density which is approximately $\rho = 0.16 \text{ fm}^{-3}$.

We notice that the minimum takes a value which is different from the one we derived thanks to the relativistic approach.

The curve, however, maintains the trend we expect for the binding energy

per nucleon.

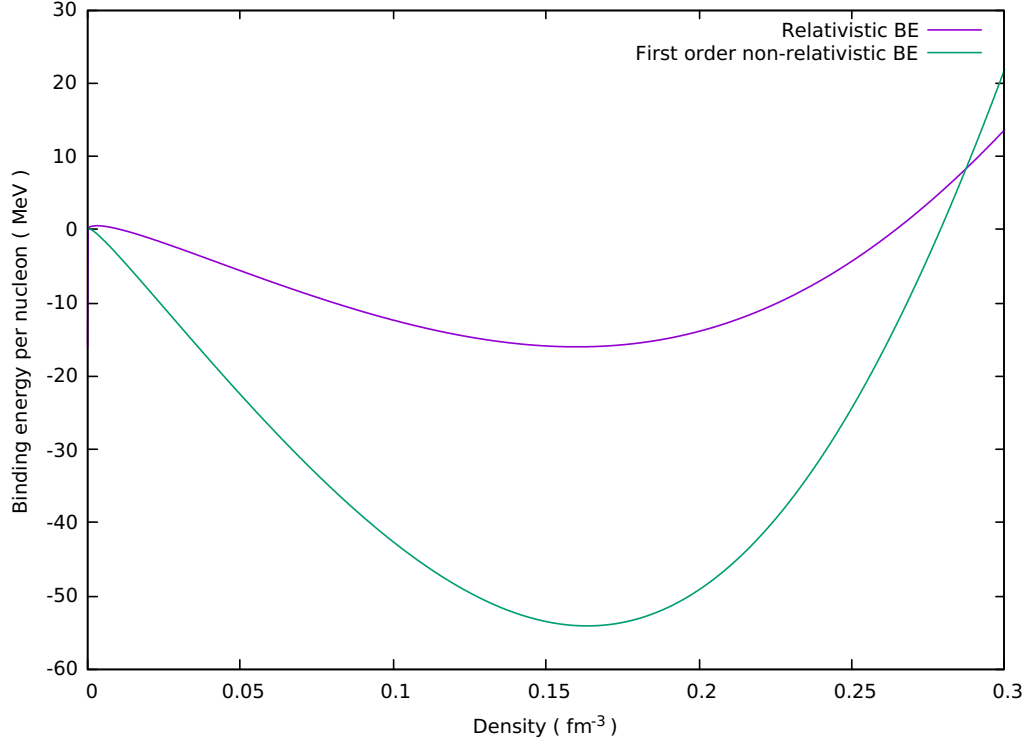


Figure 5.7: Non relativistic binding energy as a function of ρ compared to the relativistic one

Energy density In Chapter 2, we defined the binding energy as

$$\frac{E}{A} = \frac{e}{\rho} - M \quad (5.13)$$

By inverting this equation we can find the expression of the energy density as a function of the non-relativistic binding energy:

$$e = \rho \left(\frac{E}{A} + M \right) \quad (5.14)$$

We evaluate this quantity for $\rho = 0, 16 \text{ fm}^{-3}$ and we find $e = 141.587 \text{ MeVfm}^{-3}$. Looking at the graph we find a trend similar to the one we had in the relativistic model.

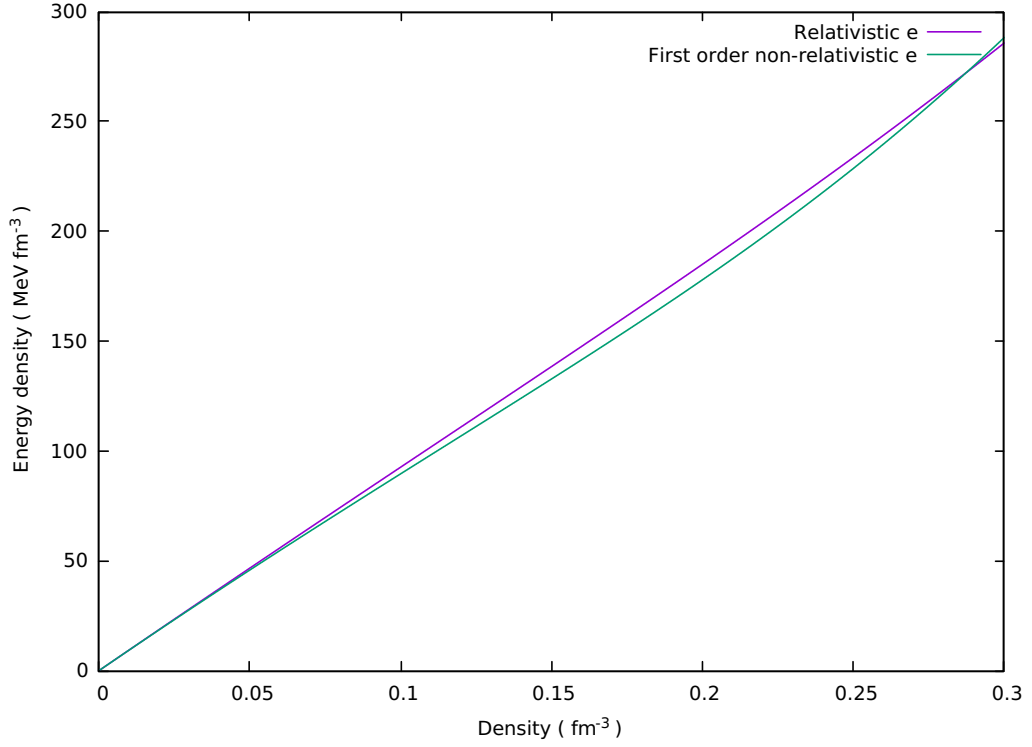


Figure 5.8: Non relativistic energy density as a function of ρ compared to the relativistic one

Pressure For the non-relativistic model we look at the thermodynamical pressure as defined in the previous Section. First of all we present the value of p for $\rho = 0, 16\text{fm}^{-3}$: $p = -1.41237 \text{ MeVfm}^{-3}$.

This means that the saturation density is not the same we imposed in the relativistic approach. Thanks to numerical calculations one can find the new saturation density: $\rho_{sat} = 0.1635\text{fm}^{-3}$. For this value of density the pressure is $p = 0.0049\text{MeVfm}^{-3}$, which is different from zero due to numerical accuracy.

Now we look at the graph in figure 5.9. We notice that the trend is like the one we observed in the relativistic model. Again, we notice that the pressure takes also negative values.

The incompressibility K here takes the value:

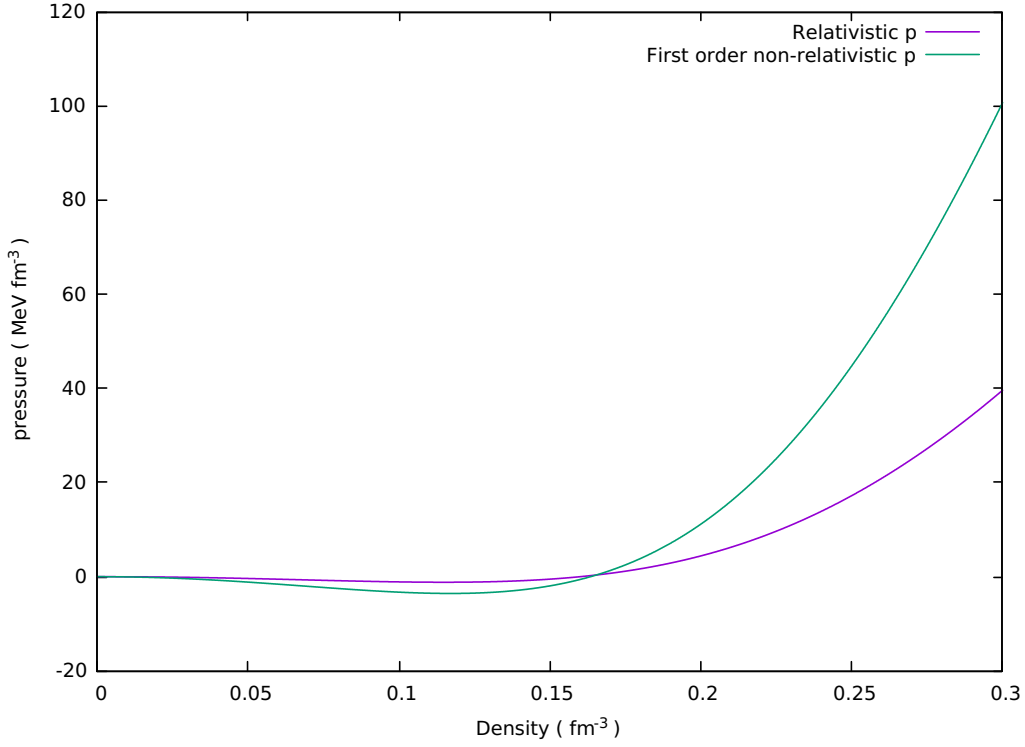


Figure 5.9: Non relativistic pressure as a function of ρ compared to the relativistic one

$$K = 1634.81 \text{ MeV} \quad (5.15)$$

The value of the incompressibility is much higher than the one we found thanks to the relativistic approach. This can be due to the simplifications we made to derive the non-relativistic results.

5.2.2 Second order reduction

The Hamiltonian of the system up to the second order reads:

$$H_{eff}^{(class)} = \mathcal{S} + \mathcal{V} + \frac{p^2}{2M^*} + \frac{p^4}{(2M^*)^3} \quad (5.16)$$

We present here the same quantities we found for the first order reduction, and we try to understand if this Hamiltonian results in a better approxima-

tion of the relativistic model.

Binding energy The binding energy per nucleon of the system can be written as:

$$\frac{E}{A} = \mathcal{S} + \mathcal{V} + \frac{3}{5} \frac{k_F^2}{2M^*} + \frac{3}{7} \frac{k_F^4}{(2M^*)^3} \quad (5.17)$$

For $\rho = 0.16 \text{ fm}^{-3}$ we find $\frac{E}{A} = -49.626 \text{ MeV}$. We notice that, thanks to the second order corrections, the binding energy is higher than the one we found with the first order reduction alone. However, it still is very different from the one we imposed in the relativistic approach.

The other values for the binding energy are given in figure 5.10.

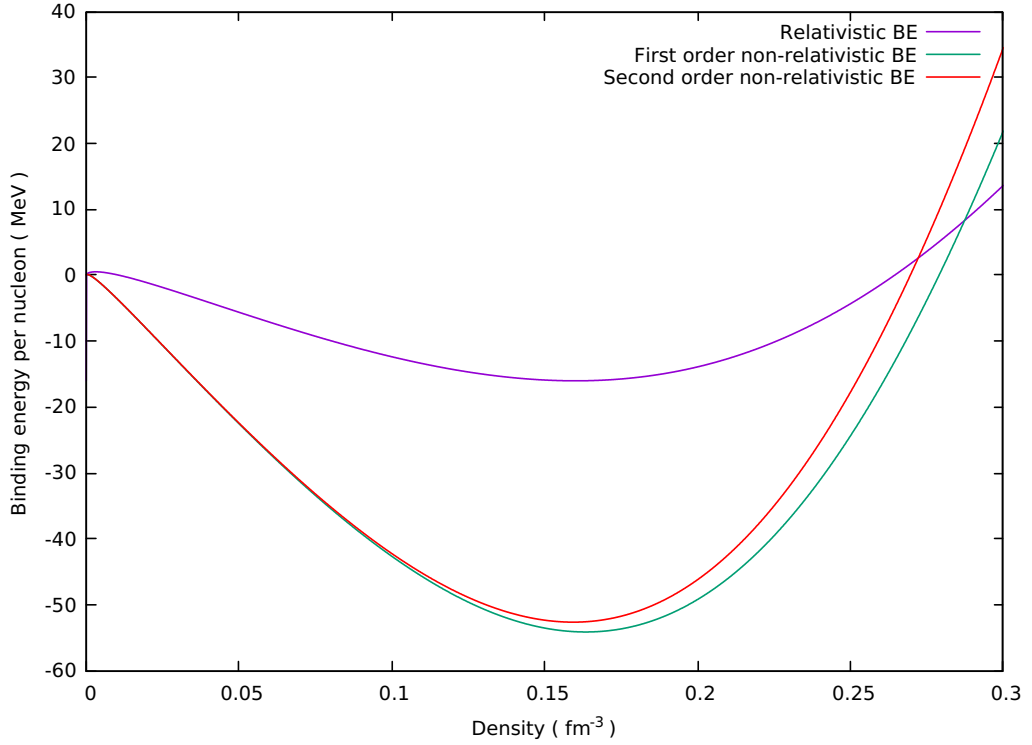


Figure 5.10: Non relativistic binding energy up to the second order as a function of ρ compared to the first order and the relativistic ones

The graph shows the trend we expect for the binding energy per nucleon.

We see, however, that for $\rho = 0.16 \text{ fm}^{-3}$ we do not have exactly the minimum, in fact, we can find it for $\rho = 0.0.1594 \text{ fm}^{-3}$.

Energy density and pressure Energy density and pressure have been found in the same way we did in the first order reduction.

For $\rho = 0.1594 \text{ fm}^{-3}$ we find $e = 141.287 \text{ MeV fm}^{-3}$ and $p = 0.008 \text{ MeV fm}^{-3}$.

The pressure is not exactly zero due to numerical accuracy.

The trends of the energy density and the pressure are shown in figure 5.11 and 5.12.

In this case we find the incompressibility K to be

$$K = 1610.7 \text{ MeV}$$

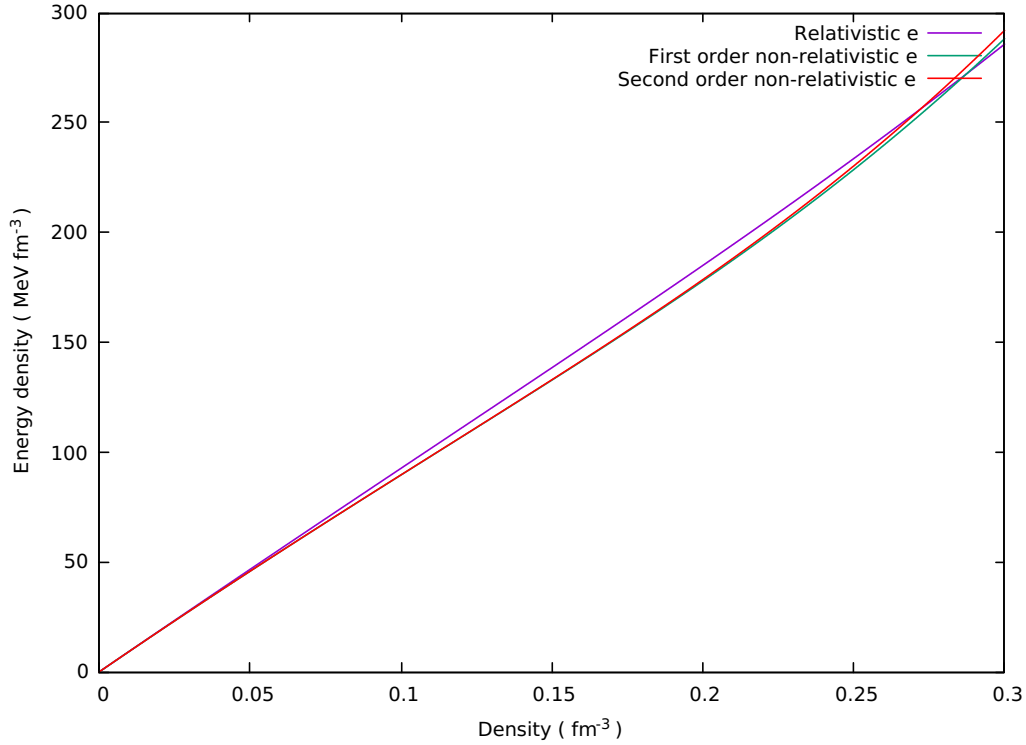


Figure 5.11: Non relativistic energy up to the second order density as a function of ρ compared to the first order and the relativistic ones

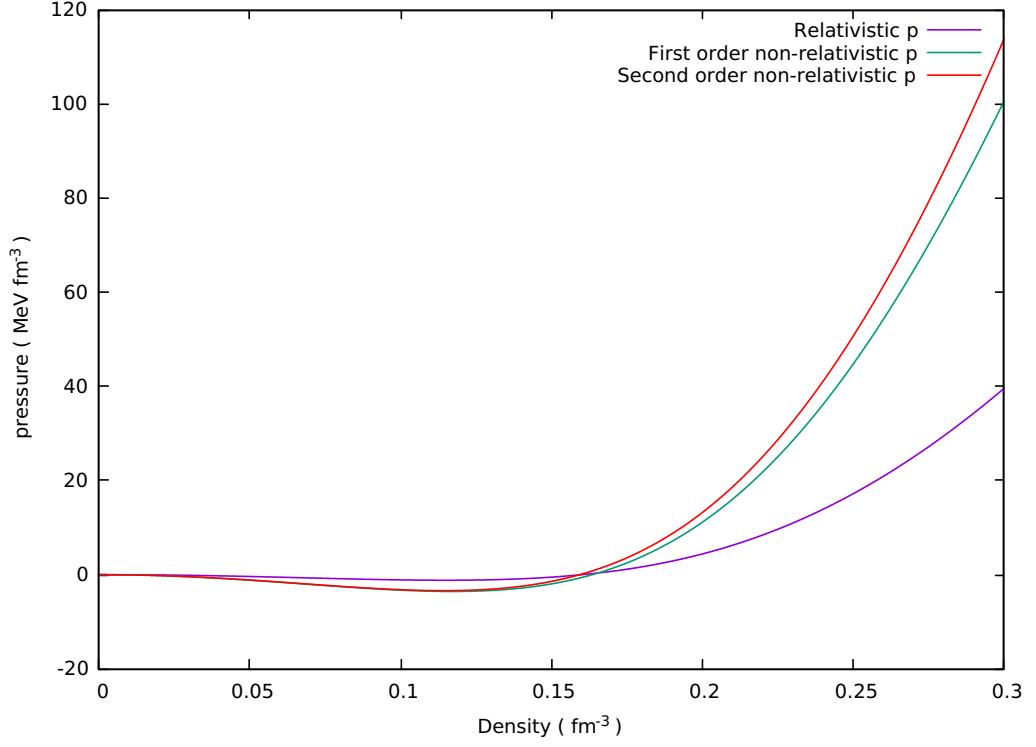


Figure 5.12: Non relativistic pressure up to the second order as a function of ρ compared to the first order and the relativistic ones

which is lower than the one we found for the first order reduction, but still higher than the one we found thanks to the relativistic approach.

5.3 Comments

As we saw thanks to graphs 5.1 to 5.6, the relativistic approach gives all the results we were expecting from the model. We fixed a minimum of the binding energy for $\rho = 0.16 \text{ fm}^{-3}$ and we find the thermodynamic pressure to be equal to the one we found thanks to the energy momentum tensor. As we said before, this is a very strong test of the model.

When we study the non-relativistic reduction (graphs from 5.7 to 5.9), however, we see that the saturation density changes, together with the minimum of the binding energy and the trend of the pressure. This can be due to the assumptions we made while deriving the model.

This approximation, thus, may not be good enough to describe the system.

Another indicator of this fact is the incompressibility value (see equation (5.15)), which is a lot higher than the one we found with the relativistic approach (almost three times bigger).

If we look at the curves, however, we can see that, for example, the binding energy maintains the trend we expect: the figure, in fact, shows a minimum for a value of $\rho \sim 0.16 \text{ fm}^{-3}$ and we have a range of densities in which the energy is negative. The minimum is approximately three times the one we find thanks to the relativistic description of the model. The energy density trend is similar to the one we found with the relativistic approach.

Chapter 6

Conclusions

With this work we studied the nuclear equation of state.

At first, we studied it in a relativistic framework: we wrote the Lagrangian, found the equations of motion and studied the problem in the Hartree mean field approximation. We studied physical quantities such as the Dirac effective mass, the Fermi energy of nucleons inside nuclear medium, the energy density and pressure and the binding energy per nucleon.

We derived numerically quantities such as the scalar density and we have shown the trend of the physical quantities we were interested in. We noticed that the relativistic model used here gives the results one expects when tries to describe the nuclear problem, except for the incompressibility pointing to the fact that the model should be improved.

Then we performed a non-relativistic reduction of the model we studied. We derived an effective Hamiltonian and studied again the energy density, the pressure and the binding energy.

The results show that the non-relativistic reduction to first order may not be good enough to describe the problem. In fact, the minimum value of the binding energy changes.

On the other hand, we noticed that the saturation density value doesn't change too much.

Thus, we decided to perform the second order expansion. We notice that the corrections we have if we keep all terms up to order v^4 give a value for the

binding energy per nucleon which is less negative than the one found thanks to the first order reduction. However, the value still is very different from the one we fixed in the relativistic framework. Also, the saturation density slightly changes again. We also notice that the incompressibility values we find thanks to the non-relativistic reduction are very different from the one we found with the relativistic description.

The best way to understand if the non-relativistic reduction of the model can describe the system of infinite symmetric nuclear matter is to proceed with the expansion of the model to higher orders in v . One can try to understand if the higher orders corrections give results which are more similar to the ones we have in the relativistic framework.

Also, one can try to improve the incompressibility value by considering the coupling constants Γ_i ($i = \sigma, \omega, \rho, \delta$) as density dependent, or by introducing non-linear couplings of the fields in the relativistic Lagrangian.

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